

Structure of the Star with Ideal Gases

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In this paper, we provide a simplified stellar structure model for ideal gases, in which the particles are only driven by gravity. According to the model, the structural information of the star can be roughly solved by the total mass and radius of a star. To get more accurate results, the model should be modified by introducing other interaction among particles and rotation of the star.

Key Words: *equation of state, thermodynamics, stellar structure, neutron star*

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I. SOME PHENOMENA INSIDE A STAR

The spherical symmetric metric for a static star is described by Schwarzschild metric

$$g_{\mu\nu} = \text{diag} (b(r), -a(r), -r^2, -r^2 \sin^2 \theta) . \quad (1.1)$$

For the energy momentum tensor of perfect fluid

$$T_{\mu\nu} = (\rho + P)U_\mu U_\nu - P g_{\mu\nu} = \text{diag} (b\rho, aP, r^2 P, r^2 \sin^2 \theta P) , \quad (1.2)$$

where $\rho(r), P(r)$ are proper mass energy density and pressure, $U_\mu = (\sqrt{b}, 0, 0, 0)$, we have the independent equations as follow

$$\left(\frac{r}{a}\right)' = 1 - 8\pi G \rho r^2, \quad (G_{00} = -8\pi G T_{00}), \quad (1.3)$$

$$\frac{b'}{b} = \frac{a-1}{r} + 8\pi G P a r, \quad (G_{11} = -8\pi G T_{11}), \quad (1.4)$$

$$P' = -(\rho + P)\frac{b'}{2b}, \quad (T^{1\nu}_{;\nu} = 0). \quad (1.5)$$

To determine the stellar structure of an irrotational star, we solve these equations. However (1.3)-(1.4) is not a closed system, the solution depends on an equation of state(EOS)

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$P = P(\rho)$. For polytropes, we take $P = P_0 \rho^\gamma$ [1]. For the compact stars, there are a lot of EOS derived from particle models[2]-[7], which provide the structural information and parameters such as the maximum mass for neutron stars. However, in these EOS the boosting effects of the gravity on particles seem to be overlooked or adopted unconsciously. As shown in [8], the static equilibrium equation (1.5) is insufficient to describe such dynamical effects on fluid, and a Gibbs' type law should be included. In this paper, according to this Gibbs' type law, we derive the stellar structure model, which shows that the boosting effects of gravity play an important role to the stellar structure.

Before expanding the model, we make a few simple calculations and examine the behavior of the metric and particles inside a star to get some intuition. The first phenomenon is that, the temporal singularity and spatial singularity occur at different time and place if the spacetime becomes singular, and the temporal one seems to occur firstly.

Denoting the mass distribution by

$$M(r) = 4\pi G \int_0^r \rho r^2 dr, \quad \mathcal{R} = 2M(r). \quad (1.6)$$

Then by (1.3) we have solution

$$a = \begin{cases} \left(1 - \frac{\mathcal{R}(r)}{r}\right)^{-1}, & \text{if } r < R, \\ \left(1 - \frac{R_s}{r}\right)^{-1}, & \text{if } r \geq R, \end{cases} \quad (1.7)$$

where R is the radius of the star, and the Schwarzschild radius becomes

$$R_s = 2M(R) = \mathcal{R}(R) = 8\pi G \int_0^R \rho r^2 dr. \quad (1.8)$$

For any normal star with $R > R_s$. From the above solution we learn $a(r)$ is a continuous function and

$$a \geq 1, \quad a(0) = \lim_{r \rightarrow \infty} a = 1, \quad a_{\max} = a(r_m), \quad (0 < r_m \leq R). \quad (1.9)$$

So the spatial singularity $a \rightarrow \infty$ does not appear at the center of the star when the singularity begins to form.

On the other hand, by (1.4) and $P \geq 0$, we find $b'(r)$ is a continuous function satisfying

$$b'(0) = 0, \quad b'(r) > 0, \quad (\forall r > 0), \quad b = 1 - \frac{R_s}{r}, \quad (r \geq R). \quad (1.10)$$

(1.9) shows $b(r)$ is a monotone increasing function of r with smoothness at least $C^1([0, \infty))$. Consequently, the temporal singularity $b \rightarrow 0$ should take place at the center.

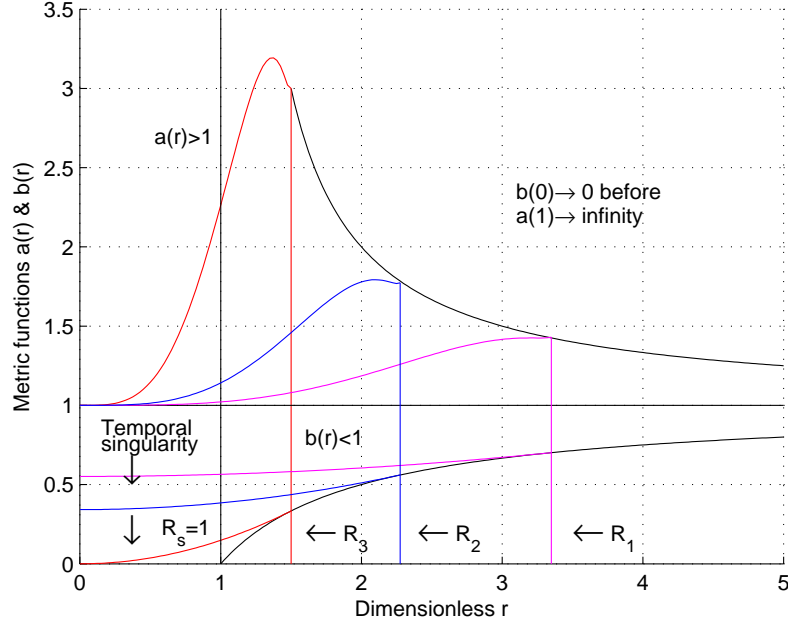


FIG. 1: The trends of (a, b) as $R \rightarrow R_s$, which show the spatial singularity does not occur at the center, but temporal singularity may occur at the center before $a \rightarrow \infty$

The trends of $a(r)$, $b(r)$ are shown in FIG.1, where we take the Schwarzschild radius $R_s = 1$ as length unit. From FIG.1 and some simplified calculations, we find $b(0) \rightarrow 0$ seems to appear before the space becomes singular $a \rightarrow \infty$.

The second phenomenon is that, the particles near the center of the star are unbalanced, and violent explosion takes place inside the star before the temporal singularity occurs. When $b(0) \rightarrow 0$, by (1.10) we have

$$b \rightarrow b_0 r^\alpha, \quad (\alpha > 1), \quad \text{if } r \ll R. \quad (1.11)$$

Substituting it into (1.5), we find

$$-P' \rightarrow (\rho + P) \frac{\alpha}{2r} \rightarrow +\infty, \quad (r \rightarrow 0). \quad (1.12)$$

According to fluid mechanics, $-\partial_r P$ corresponds to the radial boosting force, so (1.12) means violent explosion.

More clearly, we examine the motion of a particle inside the star. Solving the geodesic in the orthogonal subspace (t, r, θ) , we get[9, 10]

$$\dot{t} = \frac{1}{C_1 b}, \quad \dot{\theta} = \frac{C_2}{r^2}, \quad \dot{\varphi} = 0, \quad \dot{r}^2 = \frac{1}{r} \left(\frac{1}{C_1^2 b} - \frac{C_2^2}{r^2} - 1 \right), \quad (1.13)$$

where C_1, C_2 are constants. The normal velocity of the particle is given by

$$v_r^2 = \frac{adr^2}{bdt^2} = 1 - C_1^2 b \left(1 + \frac{C_2^2}{r^2} \right), \quad v_\theta^2 = \frac{r^2 d\theta^2}{bdt^2} = \frac{C_1^2 C_2^2 b}{r^2}, \quad v_\varphi^2 = 0. \quad (1.14)$$

The sum of the speeds provides an equality similar to the energy conservation law

$$v^2 = 1 - C_1^2 b(r), \quad \text{with} \quad v^2 \equiv v_r^2 + v_\theta^2 + v_\varphi^2. \quad (1.15)$$

(1.15) holds for all particles with $v_\varphi \neq 0$ due to the symmetry of the spacetime.

From (1.15) we learn $v \rightarrow 1$ when $b \rightarrow 0$, this means all particles escape at light velocity when the temporal singularity occurs. So instead of a final collapse, the fate of a star with heavy mass may be explosion and disintegration. The gravity of a star drives the inside particles to move rapidly and leads to high temperature. How the particles to react to the collapse of a star needs further research with dynamical models. A heuristic computation for axisymmetrical collapse is presented in [11], which reveals that the fate of a collapsing star sensitively depends on the parameters in the EOS.

II. THE EQUATIONS FOR STELLAR STRUCTURE

In this paper, we simply take the star as a ball of ideal gases, which satisfies the following assumptions:

(A1) All particles are classical ones only driven by the gravity, namely, they are characterized by 4-vector momentum p_k^μ and move along geodesic.

(A2) The collisions among particles are elastic, and then they can be ignored in statistical sense[9, 10].

(A3) The nuclear reaction and radiation are stable and slowly varying process in a normal star, in contrast with the mass energy, the energy related to this process is small noise, so we treat all photons as particles and omit the process of its generation and radiation.

These are some usual assumptions suitable for fluid stars. However, together with (1.3)-(1.4), they are enough to give us a simplified self consistent stellar structure theory. In [8], we derived the EOS for such system as follows

$$\mathcal{N} = \mathcal{N}_0 [J(J + 2\sigma)]^{\frac{3}{2}}, \quad J \equiv \frac{kT}{\bar{m}c^2}, \quad (2.1)$$

$$\rho = \mathcal{N} \left(\bar{m}c^2 + \frac{3}{2}kT \right) = \varrho [J(J + 2\sigma)]^{\frac{3}{2}} \left(1 + \frac{3}{2}J \right) c^2, \quad (2.2)$$

$$P = \mathcal{N}kT \frac{2\sigma\bar{m}c^2 + kT}{2(\sigma\bar{m}c^2 + kT)} = \varrho [J(J + 2\sigma)]^{\frac{5}{2}} \frac{c^2}{2(\sigma + J)}, \quad (2.3)$$

where \mathcal{N} is the number density of particles, $\mathcal{N}_0 = \mathcal{N}_0(\bar{m}, \sigma)$ is related to property of the particles but independent of J , $c = 2.99 \times 10^8 \text{m/s}$ the light velocity, $\sigma \doteq \frac{2}{5}$ a factor reflecting the energy distribution function, \bar{m} the mean static mass of all particles, J dimensionless temperature, which is used as independent variable, (ρ, P) are usual energy density and pressure, ϱ a mass density defined by

$$\varrho \equiv \mathcal{N}_0 \bar{m}. \quad (2.4)$$

By (2.2) and (2.3), we get the polytropic index γ is not a constant for large range of T satisfying $1 < \gamma < \frac{5}{3}$, and the velocity of sound

$$C_{\text{sound}} \equiv c \sqrt{\frac{dP}{d\rho}} = \frac{\sqrt{3}}{3} \left(\frac{c^2 J(2\sigma + J)(5\sigma^2 + 8\sigma J + 4J^2)}{(\sigma + J)^2 [2\sigma + (2 + 5\sigma)J + 4J^2]} \right)^{\frac{1}{2}} < \frac{\sqrt{3}}{3} c, \quad (2.5)$$

which shows that the EOS is regular and the causality condition holds.

Equation (1.5) can be rewritten as

$$\frac{db}{dJ} = -\frac{2b}{\rho + P} \frac{dP}{dJ}. \quad (2.6)$$

Substituting (2.2) and (2.3) into (2.6), we solve

$$b = \frac{4(R - R_s)(\sigma + J)^2}{R[2\sigma + (2 + 5\sigma)J + 4J^2]^2}, \quad (2.7)$$

where R is the radius of the star, and $R_s = 2M_{\text{tot}}$ the Schwarzschild radius. Substituting (1.7) and (2.2)-(2.7) into (1.3) and (1.4), instead of the Tolman-Oppenheimer-Volkoff equation and Lane-Emden equation[1], we get the following dimensionless equations for stellar structure,

$$\mathcal{R}'(r) = \left(\frac{r}{\chi} \right)^2 (2 + 3J)[J(J + 2\sigma)]^{\frac{3}{2}}, \quad (2.8)$$

$$J'(r) = \frac{-\xi(J)}{2(r - \mathcal{R})} \left(\left(\frac{r}{\chi} \right)^2 [J(J + 2\sigma)]^{\frac{5}{2}} + \frac{\mathcal{R}}{r}(\sigma + J) \right), \quad (2.9)$$

where χ is a constant length scale,

$$\chi = c(4\pi G \varrho)^{-\frac{1}{2}} = c(4\pi G \mathcal{N}_0 \bar{m})^{-\frac{1}{2}}, \quad (2.10)$$

$$\xi \equiv \frac{2\sigma + (2 + 5\sigma)J + 4J^2}{5\sigma^2 + 8\sigma J + 4J^2}, \quad (\xi \approx 1). \quad (2.11)$$

The exact solution to (2.8) and (2.9) seems not easy to be obtained. However, they are dimensionless equations convenient for numeric resolution. If we take $\chi = 1$ as the unit of length, the solution can be uniquely determined by the following boundary conditions

$$\mathcal{R}(0) = 0, \quad J(0) = J_0 > 0, \quad J(r) = 0, \quad (\forall r \geq R). \quad (2.12)$$

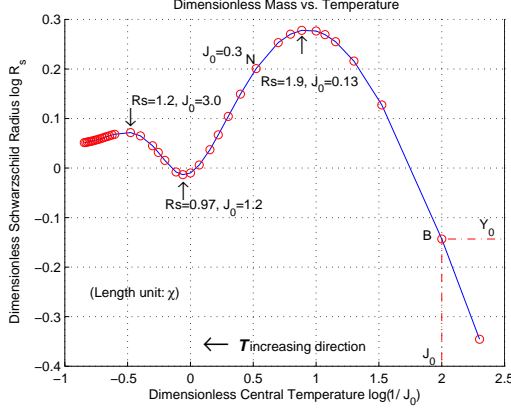


FIG. 2: Relation between mass and central temperature similar to H-R diagram, where it is concentrated by the scale χ and J .

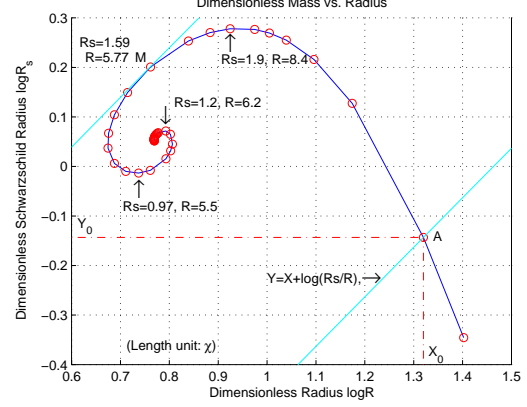


FIG. 3: Relation between mass and Radius. All structural information of a star are determined by a given radii pair (R_s, R) .

Adjusting J_0 , we get different solution. Some solutions are displayed in FIG.2-FIG.5. Usually we can easily measure the radius R and the total mass or equivalent Schwarzschild radius R_s of a star. Then the other structural parameters can be determined by this radii pair (R_s, R) . In what follows, we show how to use FIG.2 and FIG.3 to solve practical problems.

For the sun, we have the radii pair as

$$R_s = 2.96 \times 10^3 \text{ m}, \quad R_\odot = 6.96 \times 10^8 \text{ m}. \quad (2.13)$$

By $R_s/R_\odot = 4.25 \times 10^{-6}$, according to the relation shown in FIG.3, we can solve the intersection A and get the dimensionless radii pair

$$X_0 = \log(R/\chi) = 2.320, \quad Y_0 = \log(R_s/\chi) = -3.052. \quad (2.14)$$

Consequently, we have

$$\chi = 10^{-2.32} R = 3.335 \times 10^6 \text{ m}. \quad (2.15)$$

By Y_0 we get intersection B in FIG.2, and then get the central dimensionless temperature $J_0 = 1.145 \times 10^{-6}$ for the sun. Taking it as initial value we can solve (2.8) and (2.9), and then get detailed structural information for the sun.

By (2.10) and (2.15), we get

$$\varrho = \frac{c^2}{4\pi G \chi^2} = 9.640 \times 10^{12} \text{ (kg/m}^3\text{)}. \quad (2.16)$$

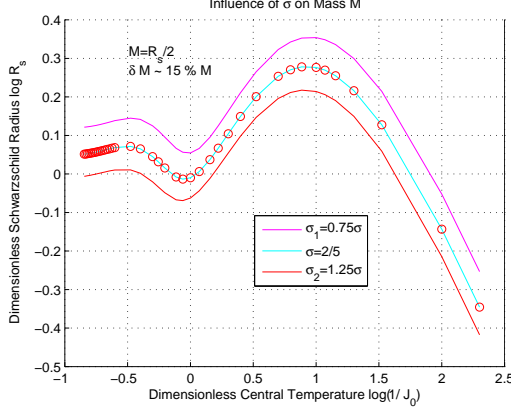


FIG. 4: The influence of energy distribution on solutions. The results are not sensitive to σ .

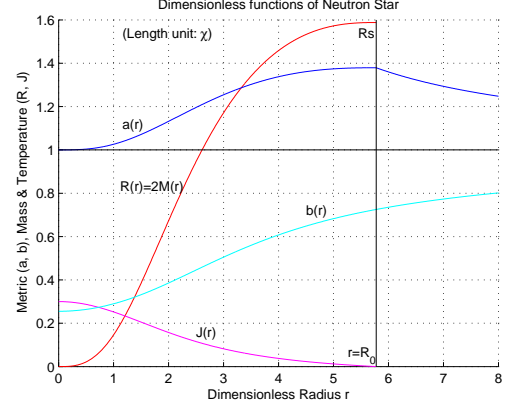


FIG. 5: Structural functions for the compactest stars. The trends are typical for all stars.

Then by (2.2) and (2.3) we solve the mass density and pressure at the center

$$\rho(0) = \varrho[J_0(J_0 + 2\sigma)]^{\frac{3}{2}} \left(1 + \frac{3}{2}J_0\right) = 8.45 \times 10^3 \text{ (kg/m}^3\text{)}, \quad (2.17)$$

$$P(0) = \frac{1}{2}\varrho[J_0(J_0 + 2\sigma)]^{\frac{5}{2}}(\sigma + J_0)^{-1}c^2 = 8.70 \times 10^8 \text{ (MPa)}. \quad (2.18)$$

The temperature depends on the mean mass \bar{m} . By (2.1), we have

$$T_c = \bar{m}c^2 J_0/k = n_p(m_p c^2 J_0/k) = 1.247 \times 10^7 n_p \text{ (K)}, \quad (2.19)$$

where $m_p = 1.673 \times 10^{-27} \text{ kg}$ is the static mass of proton, n_p is the equivalent proton number for the particles.

If the ionization in the sun is about $H^+ + N^+ + 2e^-$, then we have

$$n_p = (70\% \times 1 + 30\% \times 14)/4 = 1.23, \quad (2.20)$$

$$\bar{m} = n_p m_p = 2.06 \times 10^{-27} \text{ (kg)}, \quad (2.21)$$

$$T_c = 1.247 \times 10^7 n_p = 1.53 \times 10^7 \text{ (K)}, \quad (2.22)$$

$$\mathcal{N}_0 = c^2/(4\pi G \bar{m} \chi^2) = 4.68 \times 10^{39} \text{ (m}^{-3}\text{)}. \quad (2.23)$$

The compactest stars (with the maximum R_s/R) correspond to the points M, N in FIG.3 and FIG.2. The radii pair is $R_s : R = 1.59 : 5.77$ and the central temperature $J_0 = 0.30$. For a compact star with the solar mass M_\odot , we get the length unit and radius as

$$\chi = R_s/1.59 = 1.86 \text{ km}, \quad R = 5.77\chi = 10.7 \text{ km}. \quad (2.24)$$

Along the above procedure, we solve

$$\rho(0) = 8.50 \times 10^{18} \text{ kg/m}^3, \quad (2.25)$$

$$P(0) = 1.24 \times 10^{29} \text{ MPa}, \quad (2.26)$$

$$T(0) = 3.27 \times 10^{12} n_p \text{ K}. \quad (2.27)$$

These are typical data for a neutron star[1]-[7]. The metric functions (a, b) and the mass, temperature distributions (\mathcal{R}, J) for this star are displayed in FIG.5.

III. DISCUSSION AND CONCLUSION

The above numerical results show that, although the orbits of the particles are not simple ellipses due to collisions, the influence of the gravity still exists and leads to high temperature inside the star. The main part of the EOS may be related to gravity.

In contrast with (2.17) and (2.18), we find the central density and pressure in the sun are about one order of magnitude less than the current data

$$\rho(0) = 1.6 \times 10^5 \text{ kg/m}^3, \quad P(0) = 2.5 \times 10^{10} \text{ MPa}. \quad (3.1)$$

This difference is an unsettled serious problem, which seems to be caused by the dynamical effect of gravity.

FIG.4 shows that the solutions are not very sensitive to the concrete energy distribution or σ , so one needs not to solve specific problems via calculating complex distribution functions[8].

For photons, by the Stefan-Boltzmann's law

$$\rho = \frac{8\pi^5 (kT)^4}{15(hc)^3}, \quad (3.2)$$

we get

$$\mathcal{N}_0 \rightarrow \frac{16\pi^5}{45} \left(\frac{\bar{m}c}{h} \right)^3 = \frac{2\pi^2}{45} \left(\frac{\bar{m}c}{h} \right)^3, \quad (\bar{m} \rightarrow 0). \quad (3.3)$$

How to determine the concrete function $\mathcal{N}_0(\bar{m}, \sigma)$ is an interesting problem.

The dimensionless equations (2.8) and (2.9) simplify the relations between parameters. This function is similar to that of the similarity theory [12]. However in these equations the information of the interaction among particles is ignored, so it can not provide the critical data such as the maximum density, the largest mass. To get such data the potentials and interactive fields should be introduced to the energy momentum tensor[8].

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